Lecture #6

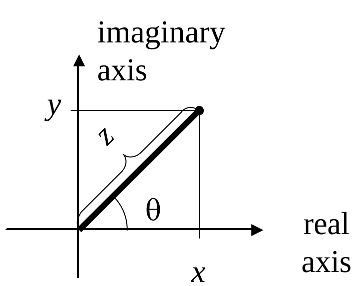
OUTLINE

- **Chap 5**
  - **Phasors**
  - **Complex Impedances**

Reading

Chap 5.1-5.4

Complex Numbers



- *x* is the real part
- *y* is the imaginary part
- *z* is the magnitude
- θ is the phase

More Complex Numbers

- Polar Coordinates: **A** = $z \angle \theta$
- Rectangular Coordinates: **A** = $x + jy$

$$x = z\cos\theta \qquad y = z\sin\theta$$

$$z = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

Summary of Phasors

- Phasor (frequency domain) is a complex number:

$$\mathbf{X} = z \angle \theta = x + jy$$

- Sinusoid is a time function:

$$x(t) = z\cos(\omega t + \theta)$$

Examples

Find the time domain representations of

$$\mathbf{X} = -1 + j2$$

$$\mathbf{V} = 104\text{V} - j60\text{V}$$

$$\mathbf{A} = -1\text{mA} - j3\text{mA}$$

Arithmetic With Complex Numbers

- To compute phasor voltages and currents, we need to be able to perform computation with complex numbers.
  - Addition
  - Subtraction
  - Multiplication
  - Division

Addition

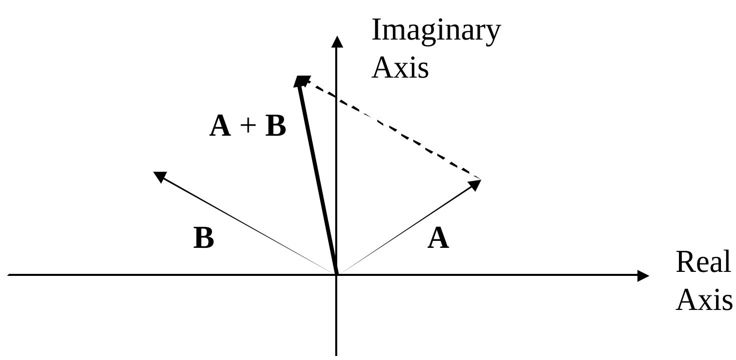
- Addition is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} + \mathbf{B} = (x + z) + j(y + w)$$

Addition



Subtraction

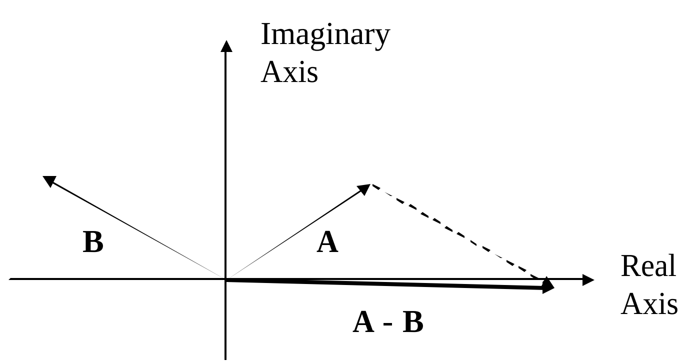
- Subtraction is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} - \mathbf{B} = (x - z) + j(y - w)$$

Subtraction



Multiplication

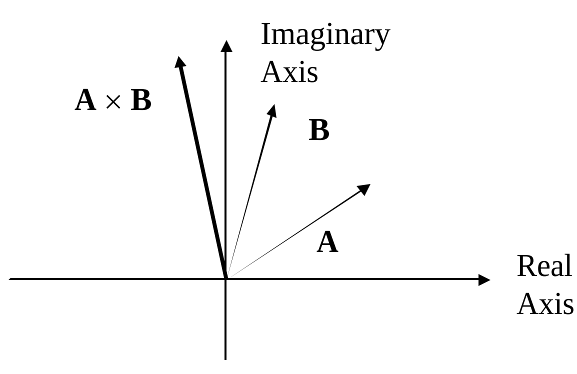
- Multiplication is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} \times \mathbf{B} = (A_M \times B_M) \angle (\theta + \phi)$$

Multiplication



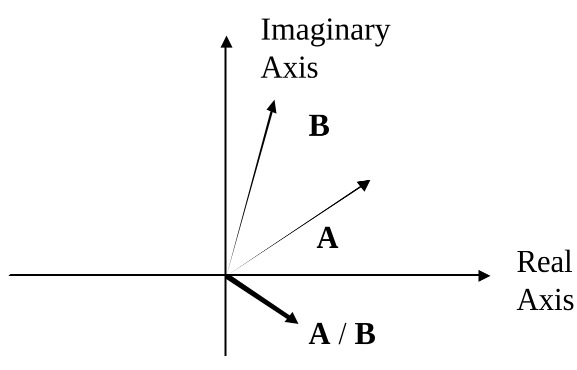
Division

- Division is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$
$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} / \mathbf{B} = (A_M / B_M) \angle (\theta - \phi)$$

Division



Phasors and Complex Exponentials

- A sinusoid can be described using a complex exponential:

  $$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

  So,

  $$v(t) = V_M \cos(\omega t + \theta) = \mathrm{Re}[e^{j\omega t + \theta}]$$

- The phasor of v(t) is given by:

  $$\mathbf{V} = V_M \angle \theta$$

Complex Exponentials

- We represent a real-valued sinusoid as the real part of a complex exponential.
- Complex exponentials provide the link between time functions and phasors.
- Complex exponentials make solving for AC steady state an algebraic problem.

Complex Exponentials

- A complex number $A = z \angle \theta$ can be represented as

  $$\boldsymbol{A} = z \angle \theta = z\, e^{j\theta} = z \cos\theta + j\, z \sin\theta$$

  DO NOT CONFUSE A SINE FUNCTION WITH A COMPLEX NUMBER!

- A What do you get when you multiply A and $e^{j\omega t}$ and find the real part?

Complex Exponentials

$$\boldsymbol{A}e^{j\omega t} = z\, e^{j\theta}\, e^{j\omega t} = z\, e^{j(\omega t+\theta)}$$

$$z\, e^{j(\omega t+\theta)} = z \cos(\omega t+\theta) + j\, z \sin(\omega t+\theta)$$

$$\mathrm{Re}[\boldsymbol{A}e^{j\omega t}] = z \cos(\omega t+\theta)$$

Sinusoids, Complex Exponentials, and Phasors

- Sinusoid:

$$z \cos(\omega t+\theta)$$

- Complex exponential:

$$\boldsymbol{A}e^{j\omega t} = z\, e^{j(\omega t+\theta)}$$

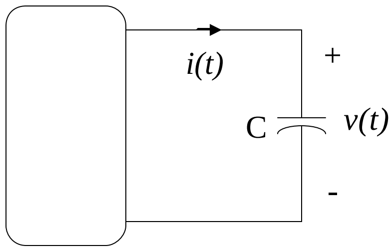
- Phasor:

$$\mathbf{V} = z \angle \theta$$

Phasor Relationships for Circuit Elements

- Phasors allow us to express current-voltage relationships for inductors and capacitors much like we express the current-voltage relationship for a resistor.
- A complex exponential is the mathematical tool needed to obtain this relationship.

I-V Relationship for a Capacitor



$$i(t) = C\frac{dv(t)}{dt}$$

Suppose that *v(t)* is a sinusoid:

$$v(t) = V_M \cos(\omega t + \theta)$$

Find *i(t)*.

Computing the Current

$$i(t) = C\frac{dv(t)}{dt} = -CV_M\omega\sin(\omega t + \theta)$$

$$i(t) = \mathrm{Re}[\,j\omega CV_M e^{j\omega t + j\theta}\,] = \mathrm{Re}[\,j\omega Cv(t)]$$

We can write:

$$i(t) = \mathrm{Re}[\frac{1}{Z_C}v(t)]$$ where $Z_C = 1/j\omega C$

Phasor Relationship

- Represent *v(t)* and *i(t)* as phasors:

$$\mathbf{V} = V_M \angle \theta$$

$$\mathbf{I} = j\omega C\, \mathbf{V}$$

- The derivative in the relationship between *v(t)* and *i(t)* becomes a multiplication by $j\omega$ in the relationship between **V** and **I**.

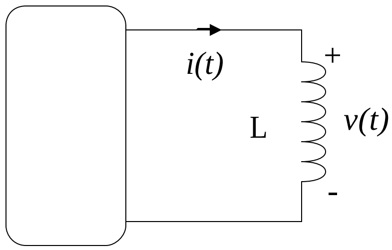
Example

$$v(t) = 120\text{V} \cos(377t + 30°)$$

$$C = 2\mu\text{F}$$

- What is **V**?
- What is **I**?
- What is *i(t)*?

I-V Relationship for an Inductor



$$v(t) = L\frac{di(t)}{dt}$$

$\mathbf{V} = j\omega L\ \mathbf{I}$

$\mathbf{Z_L} = j\omega L$

Example

$i(t) = 1\mu A \cos(2\pi\ 9.15\ 10^7 t + 30°)$

$L = 1\mu H$

- What is **I**?
- What is **V**?
- What is *v(t)*?

Resistor I-V relationship

$v_R = i_R R$ …………$\mathbf{V}_R = \mathbf{I}_R R$ where R is the resistance in ohms,
$\mathbf{V}_R$ = phasor voltage, $\mathbf{I}_R$ = phasor current
(boldface indicates complex quantity)

Capacitor I-V relationship

$i_C = C dv_C/dt$ ....Phasor current $\mathbf{I}_C$ = phasor voltage $\mathbf{V}_C$ / capacitive impedance $\mathbf{Z}_C$: → $\mathbf{I}_C = \mathbf{V}_C/\mathbf{Z}_C$
where $\mathbf{Z}_C = 1/j\omega C$, $j = (-1)^{1/2}$ and boldface indicates complex quantity

Inductor I-V relationship

$v_L = L di_L/dt$ ....Phasor voltage $\mathbf{V}_L$ = phasor current $\mathbf{I}_L$/ inductive impedance $\mathbf{Z}_L$ → $\mathbf{V}_L = \mathbf{I}_L \mathbf{Z}_L$
where $\mathbf{Z}_L = j\omega L$, $j = (-1)^{1/2}$ and boldface indicates complex quantity

Phasor Diagrams

- A phasor diagram is just a graph of several phasors on the complex plane (using real and imaginary axes).
- A phasor diagram helps to visualize the relationships between currents and voltages.

Impedance

- AC steady-state analysis using phasors allows us to express the relationship between current and voltage using a formula that looks likes Ohm's law:

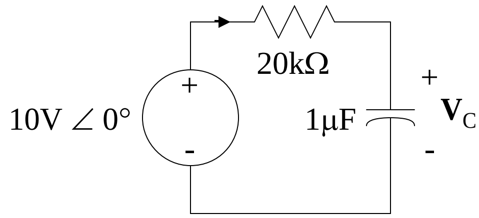
$$\mathbf{V} = \mathbf{I}\,\mathbf{Z}$$

- **Z** is called impedance.

Some Thoughts on Impedance

- Impedance depends on the frequency ω.
- Impedance is (often) a complex number.
- Impedance allows us to use the same solution techniques for AC steady state as we use for DC steady state.

Example: Single Loop Circuit

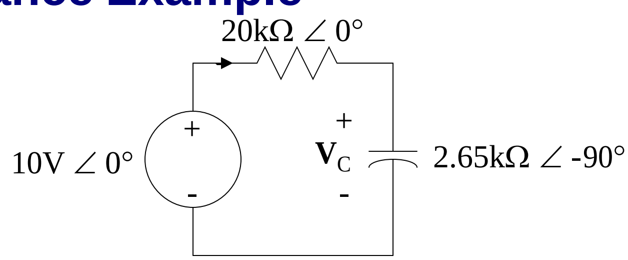


ω = 377

Find $\mathbf{V}_C$

How do we find **V**C?
First compute impedances for resistor and capacitor:
$\mathbf{Z}_R$ = 20kΩ= 20kΩ ∠ 0°
$\mathbf{Z}_C$ = 1/*j* (377 1μF) = 2.65kΩ ∠ -90°

Impedance Example

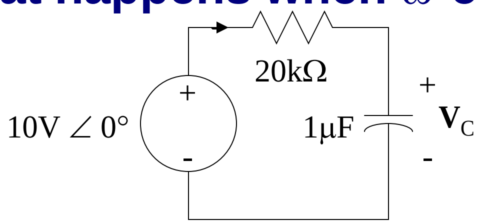


Now use the voltage divider to find $\mathbf{V}_C$:

$$\mathbf{V}_C = 10\text{V} \angle 0° \left(\frac{2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right)$$

$$\mathbf{V}_C = 1.31\text{V} \angle -82.4°$$

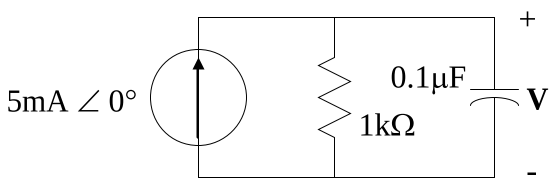
What happens when ω changes?



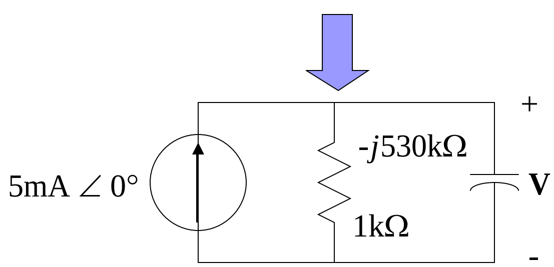
ω = 10

Find $\mathbf{V}_C$

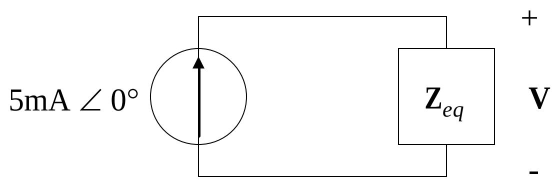
Low Pass Filter:
A Single Node-pair Circuit



Find $v(t)$ for $\omega = 2\pi\ 3000$



Find the Equivalent Impedance



$$\mathbf{Z}_{eq} = \frac{1000(-j530)}{1000 - j530} = \frac{10^3 \angle 0° \times 530 \angle -90°}{1132 \angle -27.9°}$$

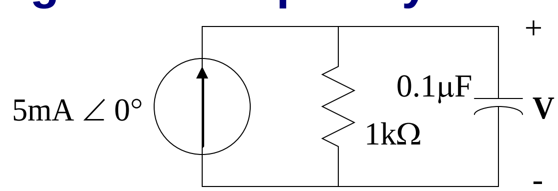
$$\mathbf{Z}_{eq} = 468.2\Omega \angle -62.1°$$

$$\mathbf{V} = \mathbf{IZ}_{eq} = 5\text{mA}\angle 0° \times 468.2\Omega \angle -62.1°$$

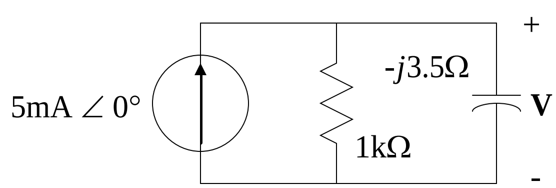
$$\mathbf{V} = 2.34\text{V} \angle -62.1°$$

$$v(t) = 2.34\text{V}\cos(2\pi\, 3000\text{t} - 62.1°)$$

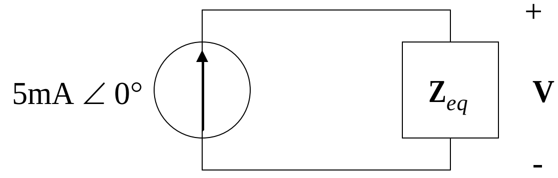
Change the Frequency



Find $v(t)$ for $\omega=2\pi$ 455000



Find an Equivalent Impedance



$$\mathbf{Z}_{eq} = \frac{1000(-j3.5)}{1000 - j3.5} = \frac{10^3\angle 0° \times 3.5\angle -90°}{1000\angle -0.2°}$$

$$\mathbf{Z}_{eq} = 3.5\Omega\angle -89.8°$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = 5\text{mA}\angle 0° \times 3.5\Omega\angle -89.8°$$

$$\mathbf{V} = 17.5\text{mV}\angle -89.8°$$

$$v(t) = 17.5\text{mV}\cos(2\pi\, 455000\text{t} - 89.8°)$$